

Finite nuclei in the reggeon “toy model”

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Received: 12 December 2016 / Accepted: 8 January 2017 / Published online: 25 January 2017
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Abstract Hadron–nucleus amplitudes at high energies are studied in the “toy” Regge model in zero transverse dimension for finite nuclei, when the standard series of fan diagrams is converted into a finite sum and loses physical sense at quite low energies. Taking into account all the loop contributions by numerical methods we find a physically meaningful amplitudes at all energies. They practically coincide with the amplitudes for infinite nuclei. A surprising result is that for finite nuclei and small enough triple pomeron coupling the infinite series of fan diagrams describes the amplitude quite well in spite of the fact that in reality the series should be cut and as such deprived of any physical sense at high energies.

1 Motivation

At high energies in the framework of the perturbative quantum chromodynamics strong interactions are mediated by the exchange of hard pomerons, which are non-local entities propagating according to the BFKL equation and splitting into two or merging from two to one with the known triple pomeron vertex. Neglecting pomeron loops and choosing the projectile to have a short range and the target to be a heavy nucleus one comes to the well-known Balitski–Kovchegov (BK) equation, which sums pomeron fan diagrams going from the projectile towards the target [1,2]. This equation with certain degrees of sophistication, including higher orders and running coupling, is widely used in applications with very positive results. However, from the start it is clear that summing all fans this equation neglects the obvious limitation that the nucleus in fact is finite, so that the number of splittings in fans is restricted. It is not clear how this restriction affects the resulting amplitude. This fact cannot be too small. For instance, taking instead of a heavy nucleus a light one, say the deuteron, one sees that the results become drastically different, since then the amplitude essen-

tially reduces to the exchange of two pomerons and grows correspondingly at high energies.

This problem is difficult to study in the framework of the full-fledged QCD pomeron theory. However, very long ago a description of the hA interaction by means of the sum of fan diagrams was proposed by Schwimmer in the reggeon field theory with a local supercritical pomeron [3]. There one easily obtains a solution of the equation, which sums all fans and is basically similar to the BK equation. Taking into account that the momenta transferred to the nucleus are small one gets for the hA amplitude at given rapidity y and fixed impact parameter b

$$\mathcal{A}(b) = \frac{g^2 AT(b)e^{\mu y}}{1 + \frac{AgT(b)\lambda}{\mu}(e^{\mu y} - 1)}. \quad (1)$$

Here μ is the pomeron intercept minus unity, assumed to be positive, $T(b)$ is the nuclear profile function normalised to unity, $\lambda > 0$ is the triple pomeron coupling with the opposite sign and g is the pomeron–nucleon coupling. This old formula possesses some nice features. In particular the Schwimmer amplitude goes to a constant value at very high energies, implying that at such energies the nucleus behaves like a (grey) disk, which more or less agrees with the later more sophisticated treatments. We shall be interested in not so much of its physical applications but rather as a tool to study our problem: how the limitations on the number of splitting coming from the actual finiteness of A change the result.

Having the explicit solution (1) this is quite trivial to see. Let

$$z = \frac{AgT(b)\lambda}{\mu}(e^{\mu y} - 1), \quad (2)$$

so that the Schwimmer amplitude is just

$$\mathcal{A}(b) = \frac{gAT(b)e^{\mu y}}{1 + z}. \quad (3)$$

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For a finite nucleus with atomic number A obviously we get just A first terms of the expansion of (3) in powers of z

$$\begin{aligned}\mathcal{A}_A(b) &= gAT(b)e^{\mu y} \sum_{n=0}^A (-z)^n \\ &= gAT(b)e^{\mu y} \frac{1 - (-z)^{A+1}}{1 + z}.\end{aligned}\quad (4)$$

Here and in the following we denote with subindex A the amplitudes which refer to finite nuclei of atomic number A , leaving the amplitude for infinite nucleus without any subindex. One immediately sees that if $z < 1$ then the series in (4) is convergent and the error in using the infinite nuclei instead of the finite (and physical) is exponentially small. On the other hand, if $z > 1$ the series (4) is divergent and the Schwimmer formula for infinite nucleus has nothing in common with the real amplitude for finite nucleus.

Condition of validity of the fan diagram amplitude $z < 1$ translates into the restriction on the highest rapidity where this amplitude has the physical meaning

$$e^{\mu y} < 1 + \frac{\mu}{AgT(b)\lambda}.\quad (5)$$

If the internucleon distance in the nucleus is R_0 then crudely estimating we have

$$g \sim R_0, \quad AT(b) = A^{1/3}/R_0^2$$

and (5) gives

$$y < \frac{1}{\mu} \ln \left(1 + A^{-1/3} \frac{\mu R_0}{\lambda} \right).\quad (6)$$

With fixed λ independent of A and large $A \gg 1$ this degenerates into

$$y < A^{-1/3} \frac{\mu R_0}{\lambda} \ll 1,\quad (7)$$

which has little physical meaning. The only possibility to have some sense for the Schwimmer amplitude for realistic nuclei is to choose λ extremely small to compensate factor $A^{-1/3}$ in (6). Otherwise this formula is just an analytic continuation of the physical amplitude having no relation to reality.

This problem is of course not a new one and is not restricted to only fan diagrams. In a simpler case of the scattering of a hadron on a nuclear target in the Glauber approximation one gets the amplitude

$$i\mathcal{A}_A(b) = (1 + ia(y)T(b))^A - 1,\quad (8)$$

where a is the forward proton–proton scattering amplitude. In the limit $A \rightarrow \infty$ and $T(b) \propto A^{-2/3}$ one gets the standard expression

$$\mathcal{A}(b) = 1 - e^{iAT(b)a(y)},\quad (9)$$

which is quite attractive, since it is explicitly unitary. However, Eq. (8) is unitary only while $|a(y)T(b)| < 1$. This is always so when the proton–proton scattering amplitude is unitary itself. However, if one takes for $a(y)$ the amplitude corresponding to the exchange of a supercritical pomeron and so rising with y as $\exp(\mu y)$ then the expression for finite nuclei loses sense for high enough y . The limiting expression for infinite nuclei preserves its unitary character but its relation to the physical amplitude becomes lost.

Observing these examples we may conclude that in both cases, fans and Glauber, the origin of the difficulty lies in the wrong behaviour of the elementary proton–proton scattering amplitude (pomeron propagator) growing exponentially with rapidity and violating unitarity. This wrong behaviour is possibly cured by inclusion of contributions from pomeron loops. We cannot be sure that this happens in the reggeon field theory, which lies at the basis of the Schwimmer equation (1). Still less is known about the behaviour of the propagator of the non-local pomeron in the perturbative QCD. However, there is a simple model where calculation of all loop contributions is possible. This is a reggeon “toy model” in the zero-dimensional transverse space. Having rapidity as the only variable it actually reduces to a sort of quantum mechanics with a non-Hermithean interaction. This theory has been extensively studied in the past [4–9] and in the limit $\lambda \rightarrow 0$ it was shown analytically that inclusion of loops makes the pomeron propagator vanish in the high-energy limit. More recently it was considered in [10, 11] and in [11] a calculational technique was elaborated which allowed one to numerically sum all contributions for arbitrary values of parameters. In this paper we apply this technique to study the behaviour of the hA amplitude in the model for finite nuclei and establish the relation between the sum of fan diagrams and the full amplitude in this case.

Note that the fan amplitude in the toy model is identical to the more physical Schwimmer amplitude (1). The simplification of neglecting the transverse space is felt only in the loop diagrams, which in the toy model are certainly different from the ones in the reggeon field theory with transverse dimensions. Still we expect that lessons known in the study of the toy model will be instructive to consider the situation in more physical theories including the perturbative QCD.

2 The toy model

In this section we briefly recapitulate the definition and properties of the toy model indispensable for our study, referring the reader either to old papers [4–9] or to the comparatively recent paper [11]. The toy model is the Regge–Gribov theory of a pomeron field $\phi(y)$ depending only on rapidity y (zero-

dimensional transverse space), which may be defined by the functional integral

$$Z = \int D\phi D\phi^\dagger e^{-S}, \quad S = \int dy \mathcal{L}, \quad (10)$$

where

$$\mathcal{L} = \frac{1}{2}(\phi^\dagger \phi_y - \phi_y^\dagger \phi) - \mu \phi^\dagger \phi + i\lambda \phi^\dagger \phi (\phi^\dagger + \phi). \quad (11)$$

Here μ is the pomeron intercept ($\alpha(0) - 1$). For the supercritical pomeron $\mu > 0$. Triple pomeron coupling constant λ is also positive. The functional integral (10) converges for $\mu < 0$ (subcritical pomeron). But in the physically interesting case that $\mu > 0$ the integral does not exist. Then in fact it only serves to introduce perturbative diagrams in the Regge–Gribov approach.

One can pass to an alternative, Hamiltonian formalism, which reproduces the perturbative diagrams but is free from the restriction $\mu < 0$. It is based on a quasi-Schrodinger equation in rapidity for the wave function $\Psi(y)$,

$$\frac{d\Psi(y)}{dy} = -H\Psi(y), \quad (12)$$

with the Hamiltonian H which can be chosen to be real

$$H = \mu uv - \lambda u(u + v)v \quad (13)$$

and is a function of two operators u and v , which are anti-Hermitean to each other,

$$u^\dagger = -v, \quad v^\dagger = -u, \quad (14)$$

and satisfy the commutation relation

$$[v, u] = -1. \quad (15)$$

The operators u and v have the meaning of creation and annihilation operators of the pomeron, respectively. The vacuum state Ψ_0 , normalised to unity, satisfies $v\Psi_0 = 0$. All other states are built from Ψ_0 by application of some number of operators u . The transition amplitude from the initial state Ψ_i at rapidity $y = 0$ to the final state Ψ_f at rapidity y is given by

$$iA_{fi} = \langle \Psi_f | \Psi_i(y) \rangle, \quad \Psi_i(y) = e^{-Hy} \Psi_i. \quad (16)$$

The amplitude A_{fi} is imaginary positive so that the matrix element on the right-hand side of (16) is negative. Some care should be taken to express the initial and final scattering states $\Psi_i(u)$ and $\Psi_f(u)$ via creation operators. We take them also to be real. Assuming that the initial state representing a heavy nucleus with $A \rightarrow \infty$ has an eikonal structure we take

$$\Psi_i(u) = (1 - e^{-g_i u}) \Psi_0, \quad (17)$$

where g_i is a positive coupling constant with the initial nucleus. It is important that the final state should be taken

not as an immediate copy of (17) (with a maybe different coupling constant) but with an additional change $u \rightarrow -u$

$$\Psi_f(u) = (1 - e^{+g_f u}) \Psi_0. \quad (18)$$

As we shall see this immediately follows from the form of the amplitude at $y = 0$. So for the scattering of two nuclei we get the amplitude in terms of purely real quantities

$$\begin{aligned} iA &= \langle (1 - e^{g_f u}) \Psi_0 | e^{-Hy} | (1 - e^{-g_i u}) \Psi_0 \rangle \\ &= \langle (1 - e^{-g_f v}) e^{-Hy} (1 - e^{-g_i u}) \rangle. \end{aligned} \quad (19)$$

In the last formula the vacuum matrix element is implied. Since $H\Psi_0 = 0$ the term independent of g_i and g_f vanishes, so that we can also write

$$\begin{aligned} iA_{fi} &= -\langle e^{-g_f v} e^{-Hy} (1 - e^{-g_i u}) \rangle \\ &= -\langle \Psi_0 | e^{-g_f v} F_i(y, u) \Psi_0 \rangle, \end{aligned} \quad (20)$$

where $F_i(y, u)$ is the operator which creates the evolved initial state. It satisfies the equation

$$\frac{\partial F_i(y, u)}{\partial y} = -H(u, v) F_i(y, u) \quad (21)$$

with the initial condition

$$F_i(0, u) = 1 - e^{-g_i u}. \quad (22)$$

The commutation relation (15) allows one to represent

$$v = -\frac{\partial}{\partial u} \quad (23)$$

and then (20) implies that to find the amplitude one has to substitute u by g_f in $F_i(y, u)$

$$iA_{fi} = -F_i(y, g_f). \quad (24)$$

At $y = 0$ this gives the initial amplitude

$$\mathcal{A} = i(1 - e^{-g_f g_i}) \quad (25)$$

in clear correspondence with the nucleus–nucleus amplitude in the so-called optical approximation. Should we take the final state without reversing the sign of u we would get the sign plus in the exponent in obvious contradiction with the optical amplitude.

Taking the complex conjugate of (20) we find

$$\begin{aligned} -iA_{fi}^* &= \langle (1 - e^{g_i v}) e^{-H^\dagger y} (1 - e^{g_f u}) \rangle \\ &= iA_{if}(\lambda \rightarrow -\lambda, g_{i(f)} \rightarrow -g_{f(i)}). \end{aligned} \quad (26)$$

Having in mind that the amplitude is pure imaginary, we see that interchanging the target and projectile leads to the overall change of sign $u \rightarrow -u$. However, this will not change the amplitude. Indeed after evolution we shall get function $F_i(y, -u)$. But the change $u \rightarrow -u$ in the final state requires that now we have to substitute u by $-g_f$ so that the result will be the same $F_i(y, g_f)$ as for the direct transition. So the

interchange of the target and projectile does not change the amplitude.

3 Numerical studies

Calculation of the scattering amplitude reduces to the solution of the differential equation in two variables, y and u

$$\frac{\partial F_i(y, u)}{\partial y} = \left(\mu u \frac{\partial}{\partial u} - \lambda u^2 \frac{\partial}{\partial u} + \lambda u \frac{\partial^2}{\partial u^2} \right) F_i(y, u), \quad (27)$$

which determines evolution in y of the function $F(y, u)$ initially given at $y = 0$: $F(y, u)|_{y=0} = F_0(u)$. Note that apart from the chosen A the amplitude depends on y and two parameters μ and λ . From the form of Eq. (27) it follows that this last three variables are combined in two: the scaled rapidity $\bar{y} = \mu y$ and ratio $\rho = \mu/\lambda$. So one can explore the whole domain of rapidities y and values of μ and λ by limiting $\bar{y} \leq \bar{y}_{\max}$ and changing values of ρ appropriately.

Equation (27) can be solved analytically only in the case when one drops the term with the second derivative, which describes fusing of pomerons. The remaining equation with only the first derivatives describes propagating pomerons and their consecutive splittings, that is, fan diagrams. Its solution can easily be obtained [11]:

$$F_0(y + z), \quad z = \frac{1}{\mu} \ln \frac{u}{u - \rho}, \quad \rho = \frac{\mu}{\lambda}.$$

For the fan amplitude one chooses the initial state to be a single pomeron $F_0(u) = g_i u$ to find

$$F^{fan}(y, u) = \frac{g_i u e^{\mu y}}{1 + \frac{u}{\rho} (e^{\mu y} - 1)}. \quad (28)$$

The amplitude itself is obtained from (28) by putting $u = g_f$ and multiplication by i (it is essentially identical to the Schwimmer amplitude (1)).

With the second derivative term included, solution of Eq. (27) gives the complete amplitude with all tree diagrams and loop diagrams taken into account. Note that for nucleus–nucleus scattering the set of tree diagrams is much wider than the set of fan diagrams. Unfortunately Eq. (27) in this case cannot be solved analytically (Note, however, some important estimates at very small λ in older papers [4–9]). So one is compelled to recur to numerical methods for the solution of Eq. (27). In [11] it was found that the most straightforward approach of evolving the initial function in rapidity by the Runge–Kutta method proved to be quite feasible, provided the step in y is small enough and correlated with the step in u . In our present calculation we find good convergence with $\Delta y = 5 \cdot 10^{-7}$, $\Delta u = 1 \cdot 10^{-2}$ and the interval in u taken as $0 < u < 20$. Further diminishing of Δy or Δu or raising the maximum value of u have been found to produce no change whatsoever.

Calculations in [11] pursued a somewhat restricted goal to only illustrate the feasibility of the numerical approach and see the limiting behaviour of the propagator and hA amplitude in the limit of very high rapidities. Here we study the A dependence of the hA amplitude having in mind finite nuclei of different atomic numbers and comparison with the results for infinite nuclei. In the standard eikonal picture the effective coupling to the nucleus grows as $A^{1/3}$, which comes from the product $AT(b)$. Accordingly we take for the nucleus $g_f = A^{1/3}$ and for the nucleon $g_i = 1$ for simplicity. The hA amplitude for the infinite nucleus is then found as explained earlier. At $y = 0$ we start from $F_0(u) = u$, evolve this function according to Eq. (27) and take the final function $F(y, u)$ at $u = g_f$.

However, the main purpose of our calculations is to find what will happen when we consider realistic nuclei with finite atomic numbers A . To pass to finite nuclei we change the eikonal amplitude for the infinite nucleus to its standard Glauber form for the finite nucleus with atomic number A :

$$1 - e^{-g_f u} \rightarrow 1 - \left(1 - \frac{g_f}{A} u\right)^A. \quad (29)$$

If we change correspondingly the final state in the matrix element for the amplitude then after we evolve the initial function $F_0(u) = u$ to the desired rapidity the amplitude will be given by

$$\mathcal{A}_A = \left(1 + \frac{g_f}{A} \frac{\partial}{\partial u}\right)^A F(y, u) \Big|_{u=0}. \quad (30)$$

For not very small values of A numerical calculations of this expression are hardly feasible due to necessity to find high-order derivatives.

So instead we use the discussed symmetry under the interchange of the projectile and target and calculate the inverse amplitude with the initial state represented by the finite nucleus and the final one by the proton, that is,

$$F_0(u) = 1 - \left(1 - \frac{g_f}{A} u\right)^A, \quad F_f(u) = -u. \quad (31)$$

Our calculations were performed for the interval $0 \leq \bar{y} \leq 5$ of the scaled rapidity for three values of $\rho = 10, 2$ and 0.5 . We recall that greater values of ρ correspond to smaller values of the triple pomeron coupling λ . Taking $\mu = 0.1$ more or less in correspondence with the soft pomeron properties our amplitudes are found at rapidities up to 50. The values of λ studied are then $\lambda = 0.01$ for $\rho = 10$, $\lambda = 0.05$ for $\rho = 2$ and $\lambda = 0.2$ for $\rho = 0.5$.

Our results for $A = 8, 27, 64$ and 125 are shown in Figs. 1, 2 and 3 for the three mentioned values of ρ , respectively. In each figure we compare the fan amplitude for the finite nucleus with a given A (1), the full amplitude with loops for the finite nucleus (2), the fan amplitude for infinite nucleus Eq. (28) with g_f given by $A^{1/3}$ (3) and finally the

full amplitude with loops calculated for the infinite nucleus (with eikonal initial function) and also with $g_f = A^{1/3}$ (4).

4 Discussion

Inspection of our numerical results in Figs. 1, 2 and 3 leads to the following conclusions.

1. As expected, the finite sum of fan diagrams corresponding to the given nucleus with a finite atomic number correctly describes the hA amplitude for very small values of the triple pomeron coupling ($\rho = 10$) up to a certain value of scaled rapidity $\bar{y} < \bar{y}_{\max}$ after which the result sharply blows up practically to infinity. The limiting rapidity \bar{y} depends on A very weakly, diminishing from 1.5 for $A = 8$ to 1.1 at $A = 125$. Of course these features can be immediately read from Eq. (3). Note that at $\rho = 0.5$ the interval of \bar{y} where finite fans make some sense is close to zero, so that the corresponding curve (1) is not visible in Fig. 3.
2. Remarkably with loop taken into account the results for finite nuclei and infinite ones practically coincide in all cases except for $A = 8$ at $\rho = 10$. Moreover, at comparatively high rapidities they are very weakly dependent on A (but strongly dependent on ρ). For instance for
3. $\rho = 2$ at $\bar{y} = 5$ the hA amplitudes for $A = 8, 27, 64$ and 125 are found to be 1.23, 1.28, 1.31 and 1.34 with the total difference less than 10%. This means that the structure of the hA amplitude with loops taken into account is due mostly to formation of loops during evolution so that the amplitude quickly forgets the initial state. In the exceptional case $A = 8$ and $\rho = 10$ the small values of λ and A evidently do not allow one to form enough loops to strongly influence the evolving amplitude.
3. At small values of the triple pomeron coupling λ ($\rho = 10$) fan diagrams with infinite number of splitting (the Schwimmer formula (1)) describe the total amplitude with loops very well. This is in spite of the fact that with a finite nucleus infinite fans seem to have nothing to do with the physical amplitude. So the analytic continuation involved in extending the validity of the finite fans to infinite ones seems to effectively take into account contributions from loops at small enough λ . This surprising result might have some bearing on the validity of the fan amplitude in general and in the perturbative QCD (BK equation) in particular.
4. With the growth of λ also infinite fans cease to describe the amplitude, which can be seen from Figs. 2 and 3. With $\rho = 0.5$ at $\bar{y} = 5$ they overestimate the amplitude by 3 orders of magnitude.

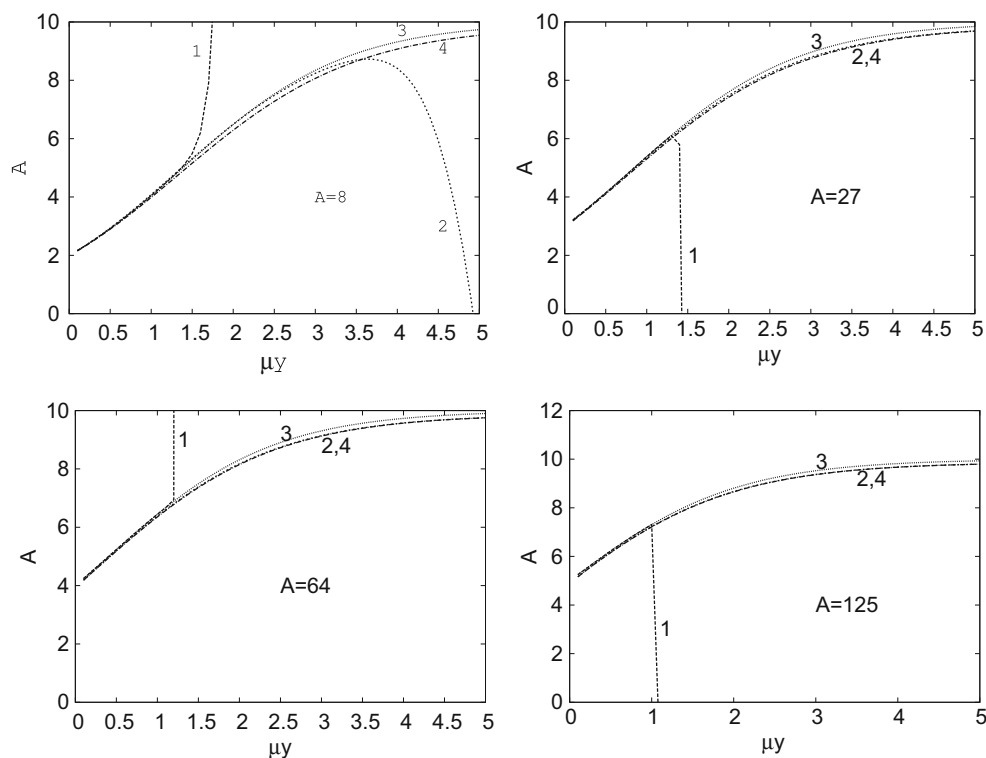


Fig. 1 hA amplitudes for $\rho = 10$. Curves show fan amplitudes for finite nuclei (1), full amplitude with loops for finite nuclei (2), infinite fan amplitudes (3), full amplitudes for infinite nuclei (4)

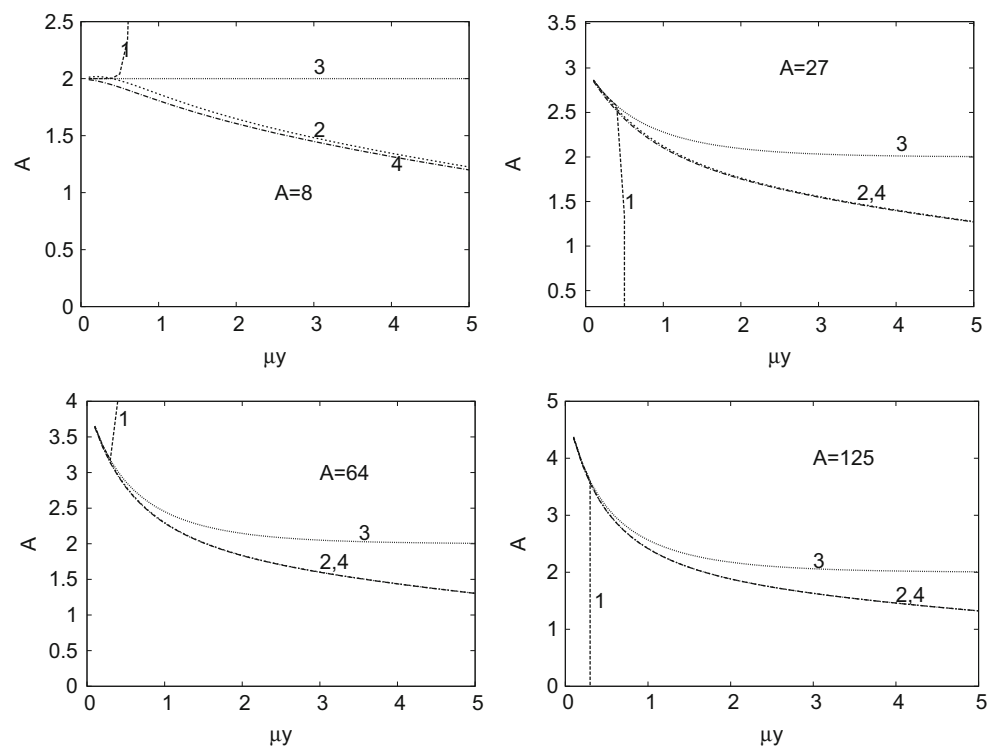


Fig. 2 Same as in Fig. 1 for $\rho = 2$

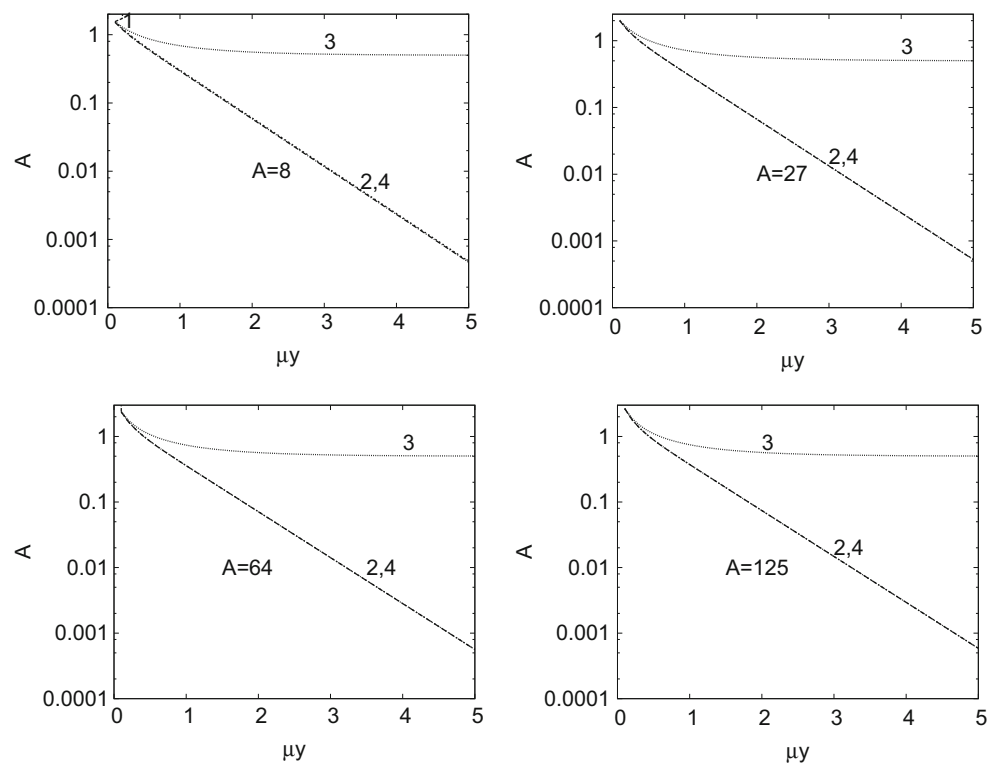


Fig. 3 Same as in Fig. 1 for $\rho = 0.5$

5 Conclusion

For realistic nuclei with finite atomic numbers A the standard infinite series of fan diagrams in the Regge theory for hA amplitude converts into a finite sum. With the elementary hadron cross-section rising at large energies this finite sum preserves physical meaning only up to a certain maximal energy, which in fact is not very large and goes down with the rise of A . Inclusion of loop diagrams may cure this situation. To study this problem we considered the toy Regge model existing in zero transverse dimensions where the loops can be taken into account by numerical methods.

Our results first show that for finite nuclei the model with contribution from loops included gives reasonable results up to very high energies. Second we discover that the found hA amplitude is practically identical to the one which corresponds to the infinite nucleus. In fact at large energies the found amplitude is weakly dependent on the initial amplitude at zero energies, so that it is formed completely from loop contributions.

Finally we found that at small enough triple pomeron coupling Λ the infinite series of fan diagrams gives a good description of the amplitude for finite nuclei in spite of the fact that in reality the series should be cut and the cut series has no physical sense at large energies. The analytic continuation in parameters involved in making the cut series to converge seems to somehow take the loop contribution into account. This conclusion, as mentioned, could explain the success of using infinite fan diagrams in the description of hA amplitudes both in the local Regge theory and the perturbative QCD.

Acknowledgements The author thanks the Bologna Physics department and INFN for hospitality. He also thanks G.P. Vacca for very interesting and constructive discussions. The author acknowledges Saint-Petersburg State University for a research Grant 11.38.223.2015. This study has also been supported by the RFBR Grant 15-02-02097.

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